



On a multivariable extension of Jacobi matrix polynomials

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ABSTRACT

The classical Jacobi matrix polynomials only for commutative matrices were first studied by Defez et al. [E. Defez, L. Jódar, A. Law. Jacobi matrix differential equation, polynomial solutions and their properties, Comput. Math. Appl. 48 (2004) 789–803]. The main aim of this paper is to construct a multivariable extension with the help of the classical Jacobi matrix polynomials (JMPs). Generating matrix functions and recurrence relations satisfied by these multivariable matrix polynomials are derived. Furthermore, general families of multilinear and multilateral generating matrix functions are obtained and their applications are presented.

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1. Introduction

Special matrix functions seen on statistics, Lie group theory and number theory are well known in [1–3]. Recently, the classical orthogonal polynomials have been extended to the orthogonal matrix polynomials [4–8]. Furthermore, one can see more papers concerning orthogonal matrix polynomials (see [9–12]). In [13], these polynomials are orthogonal as examples of right orthogonal matrix polynomial sequences for appropriate right matrix moment functionals of integral type. Hermite matrix polynomials have been introduced and studied in [14–17] for matrices in $\mathbb{C}^{N \times N}$ whose eigenvalues are all situated in the right open half-plane. In [18], the authors introduced and studied Jacobi matrix polynomials. Jódar and Cortés introduced and studied the hypergeometric matrix function $F(A, B; C; z)$ and the hypergeometric matrix differential equation in [19] and the explicit closed form general solution of it has been given in [20]. In [5], the authors introduced the Chebyshev matrix polynomials and gave some results with Chebyshev matrix polynomials. In [8], the authors studied a new system of matrix polynomials, namely the Gegenbauer matrix polynomials (see also [21]).

Throughout this paper, for a matrix $A \in \mathbb{C}^{N \times N}$, its spectrum is denoted by $\sigma(A)$. The two-norm of A , which will be denoted by $\|A\|$, is defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where, for a vector $y \in \mathbb{C}^N$, $\|y\|_2 = t(y^T y)^{1/2}$ is the Euclidean norm of y . I and θ will denote the identity matrix and the null matrix in $\mathbb{C}^{N \times N}$, respectively. For a matrix A in $\mathbb{C}^{N \times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A . We say that a matrix A in $\mathbb{C}^{N \times N}$ is a positive stable matrix, if $\operatorname{Re}(\lambda) > 0$ for all $\lambda \in \sigma(A)$. If A_0, A_1, \dots, A_n are elements of $\mathbb{C}^{N \times N}$ and $A_n \neq \theta$, then we call

$$P(x) = A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0$$

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a matrix polynomial of degree n in x . From [19], one can see

$$(P)_n = P(P+I)(P+2I) \cdots (P+(n-1)I); \quad n \geq 1; \quad (P)_0 = I. \quad (1.1)$$

From the relation (1.1), we see that

$$\frac{(-1)^k}{(n-k)!} I = \frac{(-nI)_k}{n!}; \quad 0 \leq k \leq n.$$

The hypergeometric matrix function $F(A, B; C; z)$ has been given in the form [19]

$$F(A, B; C; z) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{n!} [(C)_n]^{-1} z^n$$

for matrices A, B and C in $\mathbb{C}^{N \times N}$ such that $C + nI$ is invertible for all integer $n \geq 0$ and for $|z| < 1$. For any matrix A in $\mathbb{C}^{N \times N}$ the authors exploited the following relation due to [19]

$$(1-x)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} x^n, \quad |x| < 1.$$

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z , which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N \times N}$ with $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus in [22], it follows that:

$$f(A)g(A) = g(A)f(A).$$

Hence, if $B \in \mathbb{C}^{N \times N}$ is a matrix for which $\sigma(B) \subset \Omega$ and $AB = BA$ then

$$f(A)g(B) = g(B)f(A).$$

Furthermore, in [18], the reciprocal scalar Gamma function, $\Gamma^{-1}(z) = 1/\Gamma(z)$, is an entire function of the complex variable z . Thus, for any $C \in \mathbb{C}^{N \times N}$, the Riesz–Dunford functional calculus [22] shows that $\Gamma^{-1}(C)$ is well defined and is, indeed, the inverse of $\Gamma(C)$. Hence: if $C \in \mathbb{C}^{N \times N}$ is such that $C + nI$ is invertible for every integer $n \geq 0$, then

$$(C)_n = \Gamma(C + nI) \Gamma^{-1}(C).$$

The Jacobi matrix polynomials have been given as in [18] so that $P_n^{(A,B)}(x)$ for parameter matrices A and B whose eigenvalues, z , all satisfy $\operatorname{Re}(z) > -1$. For any positive integer n , the n th Jacobi matrix polynomials $P_n^{(A,B)}(x)$ are defined by

$$P_n^{(A,B)}(x) = \frac{(-1)^n}{n!} F\left(A + B + (n+1)I, -nI; B + I; \frac{1+x}{2}\right) (B + I)_n$$

or

$$P_n^{(A,B)}(x) = \frac{1}{n!} F\left(A + B + (n+1)I, -nI; A + I; \frac{1-x}{2}\right) (A + I)_n.$$

Also, these matrix polynomials satisfy the following equality:

$$\begin{aligned} & \int_{-1}^1 (1-x)^A (1+x)^B P_n^{(A,B)}(x) P_m^{(A,B)}(x) dx \\ &= \begin{cases} \frac{2^{A+B+I}}{n!} \Gamma(A+B+(2n+1)I) \Gamma^{-1}(A+B+(n+1)I) \\ \quad \times \Gamma(B+(n+1)I) \Gamma(A+(n+1)I) \Gamma^{-1}(A+B+2(n+1)I), & m = n \\ \theta, & m \neq n \end{cases} \end{aligned} \quad (1.2)$$

where A and $B \in \mathbb{C}^{N \times N}$ satisfy

$$\operatorname{Re}(z) > -1 \quad \text{for } z \in \sigma(A), \quad \operatorname{Re}(\eta) > -1 \quad \text{for } \eta \in \sigma(B), \quad AB = BA.$$

We organize the paper as follows:

In Section 2, we construct a multivariable extension of Jacobi matrix polynomials which were first studied by Defez et al. only for commutative matrices and show that these matrix polynomials are orthogonal with respect to weight matrix function. The special case of $s = 1$ in this extension gives Jacobi matrix polynomials given by Defez et al. In Section 3, generating matrix functions are obtained for the multivariable Jacobi matrix polynomials and with the help of generating matrix function, several recurrence formulas for the multivariable Jacobi matrix polynomials (MJMPs) are given. Multilinear and multilateral generating matrix functions are derived for MJMPs in Section 4. In Section 5, some applications of the results in Section 4 are presented. The special cases of $s = 1$ in Sections 3–5 give some recurrence relations, various bilinear and bilateral generating functions for the Jacobi matrix polynomials $P_n^{(A,B)}(x)$.

2. Multivariable extension of Jacobi matrix polynomials

With the help of the Jacobi matrix polynomials, a systematic investigation of a multivariable extension of the Jacobi matrix polynomials $P_n^{(A,B)}(x)$ is defined by

$$P_n^{(A,B)}(\mathbf{x}) = P_{n_1, \dots, n_s}^{(A_1, \dots, A_s; B_1, \dots, B_s)}(x_1, \dots, x_s) = P_{n_1}^{(A_1, B_1)}(x_1), \dots, P_{n_s}^{(A_s, B_s)}(x_s) \quad (2.1)$$

where $\mathbf{x} = (x_1, \dots, x_s)$, $\mathbf{A} = (A_1, \dots, A_s)$, $\mathbf{B} = (B_1, \dots, B_s)$, A_i and B_i are matrices in $\mathbb{C}^{N \times N}$ satisfying the spectral conditions $\operatorname{Re}(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $\operatorname{Re}(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$ for $1 \leq i \leq s$ and $\mathbf{n} = n_1 + \dots + n_s$; $n_1, \dots, n_s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$ and also all matrices are commutative. The multivariable Jacobi matrix polynomials $P_n^{(A,B)}(\mathbf{x})$ (MJMPs) are orthogonal with respect to the weight matrix function

$$\begin{aligned} \omega(\mathbf{x}, \mathbf{A}, \mathbf{B}) &= \omega(x_1, \dots, x_s; A_1, \dots, A_s; B_1, \dots, B_s) \\ &= \omega_1(x_1, A_1, B_1) \dots \omega_s(x_s, A_s, B_s) \\ &= (1-x_1)^{A_1} (1+x_1)^{B_1} \dots (1-x_s)^{A_s} (1+x_s)^{B_s} \end{aligned}$$

over the domain

$$\Omega = \{(x_1, \dots, x_s) : -1 < x_i < 1; i = 1, 2, \dots, s\}.$$

In fact, by (1.2), we have

$$\begin{aligned} &\int_{\Omega} \omega(\mathbf{x}, \mathbf{A}, \mathbf{B}) P_n^{(A,B)}(\mathbf{x}) P_m^{(A,B)}(\mathbf{x}) d\mathbf{x} \\ &= \int_{-1}^1 (1-x_1)^{A_1} (1+x_1)^{B_1} P_{n_1}^{(A_1, B_1)}(x_1) P_{m_1}^{(A_1, B_1)}(x_1) dx_1 \\ &\quad \times \dots \times \int_{-1}^1 (1-x_s)^{A_s} (1+x_s)^{B_s} P_{n_s}^{(A_s, B_s)}(x_s) P_{m_s}^{(A_s, B_s)}(x_s) dx_s \\ &= \prod_{i=1}^s \left\{ \frac{2^{A_i+B_i+I}}{n_i!} \Gamma(A_i+B_i+(2n_i+1)I) \Gamma^{-1}(A_i+B_i+(n_i+1)I) \right. \\ &\quad \times \Gamma(B_i+(n_i+1)I) \Gamma(A_i+(n_i+1)I) \Gamma^{-1}(A_i+B_i+2(n_i+1)I) \left. \right\} \delta_{m_i, n_i} \\ &\quad (m_i, n_i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; i = 1, 2, \dots, s) \end{aligned}$$

where $d\mathbf{x} = dx_1, \dots, dx_s$ and all matrices are commutative.

Thus, the following result has been established:

Theorem 2.1. The multivariable JMPs $P_n^{(A,B)}(\mathbf{x})$ are orthogonal with respect to the weight matrix function

$$\omega(\mathbf{x}, \mathbf{A}, \mathbf{B}) = (1-x_1)^{A_1} (1+x_1)^{B_1} \dots (1-x_s)^{A_s} (1+x_s)^{B_s}$$

over the domain

$$\Omega = \{(x_1, \dots, x_s) : -1 < x_i < 1; i = 1, 2, \dots, s\}$$

where A_i and B_i are matrices in $\mathbb{C}^{N \times N}$ satisfying spectral conditions $\operatorname{Re}(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $\operatorname{Re}(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$ for $1 \leq i \leq s$ and all matrices are commutative.

3. Generating matrix functions and recurrence relations for MJMPs

In [4], it was shown that Jacobi matrix polynomials (JMPs) are generated by

$$\sum_{n=0}^{\infty} (A+B+I)_n P_n^{(A,B)}(x) [(A+I)_n]^{-1} t^n = (1-t)^{-(A+B+I)} F\left(\frac{A+B+I}{2}, \frac{A+B+2I}{2}; A+I; \frac{2t(x-1)}{(1-t)^2}\right)$$

where all eigenvalues z of the matrices A and B of the Jacobi matrix polynomials $P_n^{(A,B)}(x)$ satisfy the condition $\operatorname{Re}(z) > -1$ and $|t| < 1$. On the other hand, other generating matrix function for JMPs is as follows [4]:

$$\sum_{n=0}^{\infty} P_n^{(A,B)}(x) t^n = F_4\left(I+B, I+A; I+A, I+B; \frac{(x-1)t}{2}, \frac{(x+1)t}{2}\right)$$

where $AB = BA$. Here $F_4(A, B; C, D; x, y)$ is the matrix version of the Appell's function of two variables which is defined by

$$F_4(A, B; C, D; x, y) = \sum_{n,k=0}^{\infty} (A)_{n+k} (B)_{n+k} (D)_n^{-1} (C)_k^{-1} \frac{x^k y^n}{k!n!} (\sqrt{x} + \sqrt{y} < 1)$$

where $A, B, C, D \in \mathbb{C}^{N \times N}$ and $C + nI$ and $D + nI$ are invertible for every integer $n \geq 0$. The other one is

$$\sum_{n=0}^{\infty} (C)_n (D)_n (I + B)_n^{-1} P_n^{(A,B)}(x) (I + A)_n^{-1} t^n = F_4\left(C, D; I + A, I + B; \frac{(x-1)t}{2}, \frac{(x+1)t}{2}\right)$$

where $A, B, C, D \in \mathbb{C}^{N \times N}$ and $A + nI$ and $B + nI$ are invertible for every integer $n \geq 0$.

In this section, we obtain generating matrix functions and recurrence relations for MJMPs $P_n^{(A,B)}(\mathbf{x})$.

Using the above expressions, we can give the following results.

Theorem 3.1. Multivariable Jacobi matrix polynomials (MJMPs) $P_n^{(A,B)}(\mathbf{x})$ are generated by

$$\begin{aligned} & \sum_{n_1, \dots, n_s=0}^{\infty} (\mathbf{A} + \mathbf{B} + I)_n [(\mathbf{A} + I)_n]^{-1} P_n^{(A,B)}(\mathbf{x}) t_1^{n_1}, \dots, t_s^{n_s} \\ &= \prod_{i=1}^s (1 - t_i)^{-(A_i+B_i+I)} F\left(\frac{A_i + B_i + I}{2}, \frac{A_i + B_i + 2I}{2}; A_i + I; \frac{2t_i(x_i - 1)}{(1 - t_i)^2}\right) \end{aligned}$$

where $|t_i| < 1$, A_i and B_i are matrices in $\mathbb{C}^{N \times N}$ satisfying spectral conditions $\text{Re}(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $\text{Re}(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$ for $1 \leq i \leq s$, all matrices are commutative and

$$\prod_{i=1}^s \left\{ (A_i + B_i + I)_{n_i} [(\mathbf{A} + I)_{n_i}]^{-1} \right\} = (\mathbf{A} + \mathbf{B} + I)_n [(\mathbf{A} + I)_n]^{-1}.$$

Theorem 3.2. For the matrix polynomials MJMPs $P_n^{(A,B)}(\mathbf{x})$, we have

$$\begin{aligned} \sum_{n_1, \dots, n_s=0}^{\infty} P_n^{(A,B)}(\mathbf{x}) t_1^{n_1}, \dots, t_s^{n_s} &= \prod_{i=1}^s F_4\left(I + B_i, I + A_i; I + A_i, I + B_i; \frac{(x_i - 1)t_i}{2}, \frac{(x_i + 1)t_i}{2}\right) \\ &\times \left(\sqrt{\frac{(x_i - 1)t_i}{2}} + \sqrt{\frac{(x_i + 1)t_i}{2}} < 1; i = 1, 2, \dots, s \right) \end{aligned}$$

where A_i and B_i are matrices in $\mathbb{C}^{N \times N}$ satisfying spectral conditions $\text{Re}(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $\text{Re}(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$ for $1 \leq i \leq s$, all matrices are commutative.

Theorem 3.3. Assume that all eigenvalues λ of matrices $A_i, B_i, C_i, D_i \in \mathbb{C}^{N \times N}$ for $i = 1, 2, \dots, s$ satisfy the condition $\text{Re}(\lambda) > -1$ and all matrices are commutative. For MJMPs $P_n^{(A,B)}(\mathbf{x})$, we have the following generating matrix function

$$\sum_{n_1, \dots, n_s=0}^{\infty} (\mathbf{C})_n (\mathbf{D})_n (I + \mathbf{B})_n^{-1} (I + \mathbf{A})_n^{-1} P_n^{(A,B)}(\mathbf{x}) t_1^{n_1}, \dots, t_s^{n_s} = \prod_{i=1}^s F_4\left(C_i, D_i; I + A_i, I + B_i; \frac{(x_i - 1)t_i}{2}, \frac{(x_i + 1)t_i}{2}\right)$$

where

$$(\mathbf{C})_n (\mathbf{D})_n (I + \mathbf{B})_n^{-1} (I + \mathbf{A})_n^{-1} = \prod_{i=1}^s (C_i)_{n_i} (D_i)_{n_i} (I + B_i)_{n_i}^{-1} (I + A_i)_{n_i}^{-1}$$

and

$$\left(\sqrt{\frac{(x_i - 1)t_i}{2}} + \sqrt{\frac{(x_i + 1)t_i}{2}} < 1; i = 1, 2, \dots, s \right).$$

Theorem 3.4. For the multivariable Jacobi matrix polynomials $P_{n_1, n_2, \dots, n_s}^{(A_1, \dots, A_s; B_1, \dots, B_s)}(\mathbf{x})$, we have for $|t| < 1$

$$\sum_{n=0}^{\infty} S_n^{(A_1, \dots, A_s; B_1, \dots, B_s)}(x_1, \dots, x_s) t^n = \prod_{i=1}^s (1 - t)^{-(A_i+B_i+I)} F\left(\frac{A_i + B_i + I}{2}, \frac{A_i + B_i + 2I}{2}; A_i + I; \frac{2t(x_i - 1)}{(1 - t)^2}\right)$$

where

$$\begin{aligned} S_n^{(A_1, \dots, A_s; B_1, \dots, B_s)}(x_1, \dots, x_s) \\ = \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \cdots \sum_{n_{s-1}=0}^{n-n_1-\dots-n_{s-2}} \left\{ \mathbf{K}(n_1, \dots, n_{s-1}) P_{n-(n_1+\dots+n_{s-1}), n_1, \dots, n_{s-1}}^{(A_1, \dots, A_s; B_1, \dots, B_s)}(x_1, \dots, x_s) \mathbf{N}(n_1, \dots, n_{s-1}) \right\}; \\ \mathbf{K}(n_1, \dots, n_{s-1}) = (A_1 + B_1 + I)_{n-(n_1+\dots+n_{s-1})} (A_2 + B_2 + I)_{n_1} \cdots (A_s + B_s + I)_{n_{s-1}}, \\ \mathbf{N}(n_1, \dots, n_{s-1}) = [(A_s + I)_{n_{s-1}}]^{-1} [(A_{s-1} + I)_{n_{s-2}}]^{-1} \cdots [(A_2 + I)_{n_1}]^{-1} [(A_1 + I)_{n-(n_1+\dots+n_{s-1})}]^{-1} \end{aligned}$$

and also A_i and B_i are matrices in $\mathbb{C}^{N \times N}$ satisfying spectral conditions $\operatorname{Re}(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $\operatorname{Re}(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$ for $1 \leq i \leq s$, all matrices are commutative.

Theorem 3.5. For the matrix polynomials MJMPs $P_n^{(\mathbf{A}, \mathbf{B})}(\mathbf{x})$, we have hypergeometric matrix representations as follows:

$$P_n^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) = \prod_{i=1}^s \left\{ \frac{(-1)^{n_i}}{n_i!} F\left(A_i + B_i + (n_i + 1)I, -n_i I; B_i + I; \frac{1 + x_i}{2}\right) (B_i + I)_{n_i} \right\}$$

or

$$P_n^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) = \prod_{i=1}^s \left\{ \frac{1}{n_i!} F\left(A_i + B_i + (n_i + 1)I, -n_i I; A_i + I; \frac{1 - x_i}{2}\right) (A_i + I)_{n_i} \right\}.$$

In order to obtain some recurrence relations, we need the following lemma.

Lemma 3.6. Let a generating matrix function for $g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C})$ be

$$(1 - t_1)^{-C_1} \cdots (1 - t_s)^{-C_s} \Psi\left(\frac{-4x_1 t_1}{(1 - t_1)^2}, \dots, \frac{-4x_s t_s}{(1 - t_s)^2}\right) = \sum_{n_1, \dots, n_s=0}^{\infty} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1} \cdots t_s^{n_s}$$

where $\mathbf{x} = (x_1, \dots, x_s)$, $\mathbf{C} = (C_1, \dots, C_s)$ and $g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C})$ is a matrix polynomial of degree n_i with respect to x_i (of total degree $n = n_1 + \dots + n_s$), provided that

$$\Psi(u_1, \dots, u_s) = \Psi_1(u_1), \dots, \Psi_s(u_s); \quad u_i = \frac{-4x_i t_i}{(1 - t_i)^2}, \quad i = 1, 2, \dots, s$$

$$\Psi_i(u_i) = \sum_{n_i=0}^{\infty} \gamma_{n_i} u_i^{n_i}, \quad \gamma_0 \neq 0.$$

Then we have

$$\begin{aligned} \text{(i)} \quad x_i \frac{\partial}{\partial x_i} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) - n_i g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) \\ = -(C_i + (n_i - 1)I) g_{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) - x_i \frac{\partial}{\partial x_i} g_{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}), \quad n_i \geq 1 \end{aligned} \quad (3.1)$$

$$\begin{aligned} \text{(ii)} \quad x_i \frac{\partial}{\partial x_i} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) - n_i g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) \\ = -C_i \sum_{k=0}^{n_i-1} g_{n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) - 2x_i \sum_{k=0}^{n_i-1} \frac{\partial}{\partial x_i} g_{n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}), \quad n_i \geq 1 \end{aligned} \quad (3.2)$$

$$\text{(iii)} \quad x_i \frac{\partial}{\partial x_i} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) - n_i g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) = \sum_{k=0}^{n_i-1} (-1)^{n_i-k} (C_i + 2kl) g_{n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}), \quad n_i \geq 1 \quad (3.3)$$

where $C_i C_j = C_j C_i$ for $i, j = 1, \dots, s$.

Proof. In order to derive (3.1)–(3.3), if we differentiate

$$G(\mathbf{x}, \mathbf{t}, \mathbf{C}) = (1 - t_1)^{-C_1} \cdots (1 - t_s)^{-C_s} \Psi\left(\frac{-4x_1 t_1}{(1 - t_1)^2}, \dots, \frac{-4x_s t_s}{(1 - t_s)^2}\right)$$

with respect to x_i and t_i , we have

$$x_i(1 + t_i) \frac{\partial G}{\partial x_i} - t_i(1 - t_i) \frac{\partial G}{\partial t_i} = -C_i t_i G. \quad (3.4)$$

Eq. (3.4) can be rewritten in the forms

$$x_i \frac{\partial G}{\partial x_i} - t_i \frac{\partial G}{\partial t_i} = -C_i t_i G - t_i^2 \frac{\partial G}{\partial t_i} - x_i t_i \frac{\partial G}{\partial x_i}, \quad (3.5)$$

$$x_i \frac{\partial G}{\partial x_i} - t_i \frac{\partial G}{\partial t_i} = -\frac{C_i t_i}{1 - t_i} G - \frac{2x_i t_i}{1 - t_i} \frac{\partial G}{\partial x_i}, \quad (3.6)$$

$$x_i \frac{\partial G}{\partial x_i} - t_i \frac{\partial G}{\partial t_i} = -\frac{C_i t_i}{1 + t_i} G - \frac{2t_i^2}{1 + t_i} \frac{\partial G}{\partial t_i}. \quad (3.7)$$

Since

$$G(\mathbf{x}, \mathbf{t}, \mathbf{C}) = \sum_{n_1, \dots, n_s=0}^{\infty} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_s^{n_s},$$

(3.5) yields that

$$\begin{aligned} & \sum_{n_1, \dots, n_s=0}^{\infty} \left[x_i \frac{\partial}{\partial x_i} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) - n_i g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) \right] t_1^{n_1}, \dots, t_s^{n_s} \\ &= -C_i \sum_{n_1, \dots, n_s=0}^{\infty} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_{i-1}^{n_{i-1}} t_i^{n_i+1} t_{i+1}^{n_{i+1}}, \dots, t_s^{n_s} \\ & \quad - \sum_{n_1, \dots, n_s=0}^{\infty} n_i g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_{i-1}^{n_{i-1}} t_i^{n_i+1} t_{i+1}^{n_{i+1}}, \dots, t_s^{n_s} \\ & \quad - \sum_{n_1, \dots, n_s=0}^{\infty} x_i \frac{\partial}{\partial x_i} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_{i-1}^{n_{i-1}} t_i^{n_i+1} t_{i+1}^{n_{i+1}}, \dots, t_s^{n_s} \\ &= \sum_{\substack{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_s=0 \\ n_i=1}}^{\infty} \left\{ - (C_i + (n_i - 1)I) g_{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) \right. \\ & \quad \left. - x_i \frac{\partial}{\partial x_i} g_{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) \right\} t_1^{n_1}, \dots, t_s^{n_s} \end{aligned}$$

which leads to (3.1).

Eq. (3.6) implies that

$$\begin{aligned} & \sum_{n_1, \dots, n_s=0}^{\infty} \left[x_i \frac{\partial}{\partial x_i} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) - n_i g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) \right] t_1^{n_1}, \dots, t_s^{n_s} \\ &= -C_i \left(\sum_{k=0}^{\infty} t_i^{k+1} \right) \left(\sum_{n_1, \dots, n_s=0}^{\infty} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_s^{n_s} \right) - 2x_i \left(\sum_{k=0}^{\infty} t_i^{k+1} \right) \left(\sum_{n_1, \dots, n_s=0}^{\infty} \frac{\partial}{\partial x_i} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_s^{n_s} \right) \\ &= -C_i \sum_{n_1, \dots, n_s=0}^{\infty} \sum_{k=0}^{n_i} g_{n_1, \dots, n_{i-1}, n_i-k, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_{i-1}^{n_{i-1}} t_i^{n_i+1} t_{i+1}^{n_{i+1}}, \dots, t_s^{n_s} \\ & \quad - 2x_i \sum_{n_1, \dots, n_s=0}^{\infty} \sum_{k=0}^{n_i} \frac{\partial}{\partial x_i} g_{n_1, \dots, n_{i-1}, n_i-k, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_{i-1}^{n_{i-1}} t_i^{n_i+1} t_{i+1}^{n_{i+1}}, \dots, t_s^{n_s} \\ &= -C_i \sum_{n_1, \dots, n_s=0}^{\infty} \sum_{k=0}^{n_i} g_{n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_{i-1}^{n_{i-1}} t_i^{n_i+1} t_{i+1}^{n_{i+1}}, \dots, t_s^{n_s} \\ & \quad - 2x_i \sum_{n_1, \dots, n_s=0}^{\infty} \sum_{k=0}^{n_i} \frac{\partial}{\partial x_i} g_{n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_{i-1}^{n_{i-1}} t_i^{n_i+1} t_{i+1}^{n_{i+1}}, \dots, t_s^{n_s} \\ &= -C_i \sum_{\substack{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_s=0 \\ n_i=1}}^{\infty} \sum_{k=0}^{n_i-1} g_{n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_{i-1}^{n_{i-1}} t_i^{n_i} t_{i+1}^{n_{i+1}}, \dots, t_s^{n_s} \\ & \quad - 2x_i \sum_{\substack{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_s=0 \\ n_i=1}}^{\infty} \sum_{k=0}^{n_i-1} \frac{\partial}{\partial x_i} g_{n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_{i-1}^{n_{i-1}} t_i^{n_i} t_{i+1}^{n_{i+1}}, \dots, t_s^{n_s} \end{aligned}$$

which leads to (3.2).

From (3.7), we obtain

$$\begin{aligned}
 & \sum_{n_1, \dots, n_s=0}^{\infty} \left[x_i \frac{\partial}{\partial x_i} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) - n_i g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) \right] t_1^{n_1}, \dots, t_s^{n_s} \\
 &= -C_i \left(\sum_{k=0}^{\infty} (-1)^k t_i^{k+1} \right) \left(\sum_{n_1, \dots, n_s=0}^{\infty} g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_s^{n_s} \right) \\
 &\quad - 2 \left(\sum_{k=0}^{\infty} (-1)^k t_i^{k+1} \right) \left(\sum_{n_1, \dots, n_s=0}^{\infty} n_i g_{n_1, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_s^{n_s} \right) \\
 &= - \sum_{n_1, \dots, n_s=0}^{\infty} \sum_{k=0}^{n_i} \left\{ (-1)^k (C_i + 2(n_i - k)I) g_{n_1, \dots, n_{i-1}, n_i-k, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_{i-1}^{n_{i-1}} t_i^{n_i-k+1} t_{i+1}^{n_{i+1}}, \dots, t_s^{n_s} \right\} \\
 &= - \sum_{n_1, \dots, n_s=0}^{\infty} \sum_{k=0}^{n_i} (-1)^{n_i-k} (C_i + 2kI) g_{n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_{i-1}^{n_{i-1}} t_i^{n_i-k+1} t_{i+1}^{n_{i+1}}, \dots, t_s^{n_s} \\
 &= \sum_{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_s=0}^{\infty} \sum_{k=0}^{n_i-1} \left\{ (-1)^{n_i-k} (C_i + 2kI) g_{n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_s}(\mathbf{x}, \mathbf{C}) t_1^{n_1}, \dots, t_{i-1}^{n_{i-1}} t_i^{n_i-k} t_{i+1}^{n_{i+1}}, \dots, t_s^{n_s} \right\}
 \end{aligned}$$

which gives (3.3). \square

If we choose

$$C_i = A_i + B_i + I; \quad \gamma_{n_i} = \frac{(I + A_i + B_i)_{2n_i}}{2^{2n_i} n_i!} (I + A_i)_{n_i}^{-1} \quad (3.8)$$

in Lemma 3.6, we see that the matrix polynomial g_n is

$$\begin{aligned}
 g_n(\mathbf{v}) &= (I + \mathbf{A} + \mathbf{B})_{\mathbf{n}} P_{\mathbf{n}}^{(\mathbf{A}, \mathbf{B})} (1 - 2\mathbf{v})(I + \mathbf{A})_{\mathbf{n}}^{-1} \\
 &= \prod_{i=1}^s \left\{ (A_i + B_i + I)_{n_i} P_{n_i}^{(A_i, B_i)} (1 - 2v_i) [(A_i + I)_{n_i}]^{-1} \right\}
 \end{aligned}$$

where A_i and B_i ($i = 1, 2, \dots, s$) are commutative matrices in $\mathbb{C}^{N \times N}$. Hence, Theorem 3.1 gives following results, when put in terms of \mathbf{x} rather than \mathbf{v} . Under the light of the Lemma 3.6 and also considering (3.8), one can easily obtain the next results.

Theorem 3.7. For the matrix polynomials $P_{\mathbf{n}}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x})$, we have

$$\begin{aligned}
 \text{(i)} \quad & (x_i - 1) \left[(A_i + B_i + n_i I) \frac{\partial P_{\mathbf{n}}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x})}{\partial x_i} + \frac{\partial P_{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_s}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x})}{\partial x_i} (A_i + n_i I) \right] \\
 &= (A_i + B_i + n_i I) \left[n_i P_{\mathbf{n}}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) - P_{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_s}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) (A_i + n_i I) \right], \\
 \text{(ii)} \quad & (x_i - 1) \frac{\partial P_{\mathbf{n}}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x})}{\partial x_i} - n_i P_{\mathbf{n}}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) \\
 &= -(A_i + B_i + I)_{n_i}^{-1} \sum_{k=0}^{n_i-1} \left\{ (A_i + B_i + I)_k \left[(A_i + B_i + I) P_{n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_s}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) \right. \right. \\
 &\quad \left. \left. + 2(x_i - 1) \frac{\partial P_{n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_s}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x})}{\partial x_i} \right] (I + A_i)_k^{-1} (A_i + I)_{n_i} \right\}, \\
 \text{(iii)} \quad & (x_i - 1) \frac{\partial P_{\mathbf{n}}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x})}{\partial x_i} - n_i P_{\mathbf{n}}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) = (A_i + B_i + I)_{n_i}^{-1} \sum_{k=0}^{n_i-1} \left\{ (-1)^{n_i-k} (A_i + B_i + (2k + 1)I) \right. \\
 &\quad \left. \times (A_i + B_i + I)_k P_{n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_s}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) (I + A_i)_k^{-1} (A_i + I)_{n_i} \right\},
 \end{aligned}$$

where A_i and B_i are matrices in $\mathbb{C}^{N \times N}$ satisfying spectral conditions $\text{Re}(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $\text{Re}(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$, all matrices are commutative and $A_i + B_i + kI$ is invertible for every integer $k \geq 0$ for $1 \leq i \leq s$.

Remark 3.1. For the case of $s = 1$ in (2.1), Jacobi matrix polynomials $P_n^{(A,B)}(x)$ satisfy the following equations [4]:

$$\begin{aligned} & (x-1) \left[(A+B+nI) \frac{dP_n^{(A,B)}(x)}{dx} + \frac{dP_{n-1}^{(A,B)}(x)}{dx} (A+nI) \right] \\ &= (A+B+nI) \left[nP_n^{(A,B)}(x) - P_{n-1}^{(A,B)}(x)(A+nI) \right], \\ & (x-1) \frac{dP_n^{(A,B)}(x)}{dx} - nP_n^{(A,B)}(x) \\ &= -(A+B+I)_n^{-1} \sum_{k=0}^{n-1} \left\{ (A+B+I)_k \left[(A+B+I)P_k^{(A,B)}(x) + 2(x-1) \frac{dP_k^{(A,B)}(x)}{dx} \right] (I+A)_k^{-1} (A+I)_n \right\}, \\ & (x-1) \frac{dP_n^{(A,B)}(x)}{dx} - nP_n^{(A,B)}(x) \\ &= (A+B+I)_n^{-1} \sum_{k=0}^{n-1} \left\{ (-1)^{n-k} (A+B+(2k+1)I) (A+B+I)_k P_k^{(A,B)}(x) (I+A)_k^{-1} (A+I)_n \right\}, \end{aligned}$$

where all eigenvalues z of the matrices A and B of the Jacobi matrix polynomials $P_n^{(A,B)}(x)$ satisfy the condition $\operatorname{Re}(z) > -1$ and $A+B+nI$ is invertible for every integer $n \geq 0$.

4. Multilinear and multilateral generating matrix functions

In recent years, by making use of the familiar group-theoretic (Lie algebraic) method a certain mixed trilateral finite-series relationships have been proved for orthogonal polynomials (see, for instance, [23]). In this section, we derive several families of multilinear and multilateral generating matrix functions for the MJMPs without using Lie algebraic techniques but, with the help of the similar method as considered in [4,24].

Theorem 4.1. Corresponding to a non-vanishing function $\Omega_\mu(y_1, \dots, y_r)$ of r complex variables y_1, \dots, y_r ($r \in \mathbb{N}$) and of complex order μ , let

$$\Lambda_{\mu,v}(y_1, \dots, y_r; z) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+vk}(y_1, \dots, y_r) z^k \quad (4.1)$$

($a_k \neq 0$, $\mu, v \in \mathbb{C}$) and

$$\Theta_{\mathbf{n},p,\mu,v}(\mathbf{x}; y_1, \dots, y_r; \zeta) := \sum_{k=0}^{[n_1/p]} a_k P_{n_1-pk, n_2, \dots, n_s}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) \Omega_{\mu+vk}(y_1, \dots, y_r) \zeta^k \quad (4.2)$$

where $\mathbf{n} = n_1 + \dots + n_s$; $n_1, \dots, n_s, p \in \mathbb{N}$, $\mathbf{x} = (x_1, \dots, x_s)$, $\mathbf{A} = (A_1, \dots, A_s)$, $\mathbf{B} = (B_1, \dots, B_s)$ and A_i and B_i are matrices in $\mathbb{C}^{N \times N}$ satisfying the spectral conditions $\operatorname{Re}(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $\operatorname{Re}(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$ for $1 \leq i \leq s$, all matrices are commutative. Then we have

$$\begin{aligned} & \sum_{n_1, \dots, n_s=0}^{\infty} \Theta_{\mathbf{n},p,\mu,v} \left(\mathbf{x}; y_1, \dots, y_r; \frac{\eta}{t_1^p} \right) t_1^{n_1}, \dots, t_s^{n_s} \\ &= \prod_{i=1}^s F_4 \left(I + B_i, I + A_i; I + A_i, I + B_i; \frac{(x_i-1)t_i}{2}, \frac{(x_i+1)t_i}{2} \right) \\ & \quad \times \Lambda_{\mu,v}(y_1, \dots, y_r; \eta) \left(\sqrt{\frac{(x_i-1)t_i}{2}} + \sqrt{\frac{(x_i+1)t_i}{2}} < 1; i = 1, 2, \dots, s \right) \end{aligned} \quad (4.3)$$

provided that each member of (4.3) exists.

Proof. For convenience, let S denote the first member of the assertion (4.3) of Theorem 4.1. Then, plugging the polynomials

$$\Theta_{\mathbf{n},p,\mu,v} \left(\mathbf{x}; y_1, \dots, y_r; \frac{\eta}{t_1^p} \right)$$

from the definition (4.2) into the left-hand side of (4.3), we obtain

$$S = \sum_{n_1, \dots, n_s=0}^{\infty} \sum_{k=0}^{[n_1/p]} a_k P_{n_1-pk, n_2, \dots, n_s}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) \Omega_{\mu+vk}(y_1, \dots, y_r) \eta^k t_1^{n_1-pk} t_2^{n_2}, \dots, t_s^{n_s}. \quad (4.4)$$

Upon changing the order of summation in (4.4), if we replace n_1 by $n_1 + pk$, we can write

$$\begin{aligned} S &= \sum_{n_1, \dots, n_s=0}^{\infty} \sum_{k=0}^{\infty} a_k P_{n_1, n_2, \dots, n_s}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) \Omega_{\mu+vk}(y_1, \dots, y_r) \eta^k t_1^{n_1}, \dots, t_s^{n_s} \\ &= \left(\sum_{n_1, \dots, n_s=0}^{\infty} P_{n_1, n_2, \dots, n_s}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) t_1^{n_1}, \dots, t_s^{n_s} \right) \left(\sum_{k=0}^{\infty} a_k \Omega_{\mu+vk}(y_1, \dots, y_r) \eta^k \right) \\ &= \prod_{i=1}^s F_4 \left(I + B_i, I + A_i; I + A_i, I + B_i; \frac{(x_i - 1) t_i}{2}, \frac{(x_i + 1) t_i}{2} \right) \Lambda_{\mu, v}(y_1, \dots, y_r; \eta) \end{aligned}$$

which completes the proof of Theorem 4.1. \square

Theorem 4.2. Corresponding to an identically non-vanishing function $\Omega_{\mu}(y_1, \dots, y_r)$ of r complex variables y_1, \dots, y_r ($r \in \mathbb{N}$) and of complex order μ , let

$$\Lambda_{\mu, v}(y_1, \dots, y_r; z) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+vk}(y_1, \dots, y_r) z^k \quad (a_k \neq 0, \mu, v \in \mathbb{C}) \quad (4.5)$$

and

$$\Theta_{n, p, \mu, v}(x_1, \dots, x_s; y_1, \dots, y_r; \zeta) := \sum_{k=0}^{[n/p]} a_k S_{n-pk}^{(A_1, \dots, A_s; B_1, \dots, B_s)}(x_1, \dots, x_s) \Omega_{\mu+vk}(y_1, \dots, y_r) \zeta^k \quad (n, p \in \mathbb{N}). \quad (4.6)$$

Then we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \Theta_{n, p, \mu, v} \left(x_1, \dots, x_s; y_1, \dots, y_r; \frac{\eta}{t^p} \right) t^n \\ &= \prod_{i=1}^s (1-t)^{-(A_i+B_i+I)} F \left(\frac{A_i+B_i+I}{2}, \frac{A_i+B_i+2I}{2}; A_i+I; \frac{2t(x_i-1)}{(1-t)^2} \right) \Lambda_{\mu, v}(y_1, \dots, y_r; \eta) \end{aligned} \quad (4.7)$$

provided that each member of (4.7) exists where

$$\begin{aligned} S_n^{(A_1, \dots, A_s; B_1, \dots, B_s)}(x_1, \dots, x_s) &= \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \cdots \sum_{n_{s-1}=0}^{n-n_1-\cdots-n_{s-2}} \\ &\quad \times \left\{ \mathbf{K}(n_1, \dots, n_{s-1}) P_{n-(n_1+\cdots+n_{s-1}), n_1, \dots, n_{s-1}}^{(A_1, \dots, A_s; B_1, \dots, B_s)}(x_1, \dots, x_s) \mathbf{N}(n_1, \dots, n_{s-1}) \right\}; \end{aligned}$$

and

$$\begin{aligned} \mathbf{K}(n_1, \dots, n_{s-1}) &= (A_1 + B_1 + I)_{n-(n_1+\cdots+n_{s-1})} (A_2 + B_2 + I)_{n_1} \cdots (A_s + B_s + I)_{n_{s-1}}, \\ \mathbf{N}(n_1, \dots, n_{s-1}) &= [(A_s + I)_{n_{s-1}}]^{-1} [(A_{s-1} + I)_{n_{s-2}}]^{-1} \cdots [(A_2 + I)_{n_1}]^{-1} [(A_1 + I)_{n-(n_1+\cdots+n_{s-1})}]^{-1} \end{aligned}$$

and all matrices are commutative.

Proof. For convenience, let T denote the first member of the assertion (4.7) of Theorem 4.2. Then, upon substituting for the polynomials

$$\Theta_{n, p, \mu, v} \left(x_1, \dots, x_s; y_1, \dots, y_r; \frac{\eta}{t^p} \right)$$

from the definition (4.6) into the left-hand side of (4.7), we obtain

$$T = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k S_{n-pk}^{(A_1, \dots, A_s; B_1, \dots, B_s)}(x_1, \dots, x_s) \Omega_{\mu+vk}(y_1, \dots, y_r) \eta^k t^{n-pk}. \quad (4.8)$$

Upon inverting the order of summation in (4.8), if we replace n by $n + pk$, we can write

$$\begin{aligned} T &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k S_n^{(A_1, \dots, A_s; B_1, \dots, B_s)}(x_1, \dots, x_s) \mathcal{Q}_{\mu+vk}(y_1, \dots, y_r) \eta^k t^n \\ &= \sum_{n=0}^{\infty} S_n^{(A_1, \dots, A_s; B_1, \dots, B_s)}(x_1, \dots, x_s) t^n \sum_{k=0}^{\infty} a_k \mathcal{Q}_{\mu+vk}(y_1, \dots, y_r) \eta^k \\ &= \prod_{i=1}^s (1-t)^{-(A_i+B_i+I)} F\left(\frac{A_i+B_i+I}{2}, \frac{A_i+B_i+2I}{2}; A_i+I; \frac{2t(x_i-1)}{(1-t)^2}\right) \Lambda_{\mu,v}(y_1, \dots, y_r; \eta), \end{aligned}$$

which completes the proof of Theorem 4.2. \square

5. Further consequences

By expressing the multivariable function $\mathcal{Q}_{\mu+vk}(y_1, \dots, y_r)$, $(k \in \mathbb{N}_0, r \in \mathbb{N})$ in terms of a simpler function of one and more variables, we can give further applications of Theorems 4.1 and 4.2. For example, consider the case of $r = 1$ and $\mathcal{Q}_{\mu+vk}(y) = L_{\mu+vk}^{(C,\lambda)}(y)$ in Theorem 4.1. Here, the Laguerre matrix polynomials $L_n^{(C,\lambda)}(y)$ are defined by [7] as:

$$L_n^{(C,\lambda)}(y) = \sum_{k=0}^n \frac{(-1)^k \lambda^k}{k! (n-k)!} (C+I)_n [(C+I)_k]^{-1} y^k$$

in which C is a matrix in $\mathbb{C}^{N \times N}$, $C + nI$ is invertible for every integer $n \geq 0$ and λ is a complex number with $\operatorname{Re}(\lambda) > 0$. Notice that Laguerre matrix polynomials are generated by as follows:

$$\sum_{n=0}^{\infty} L_n^{(C,\lambda)}(y) t^n = (1-t)^{-(C+I)} \exp\left(\frac{-\lambda y t}{1-t}\right), \quad (5.1)$$

$|t| < 1$, $0 < y < \infty$. Then we obtain the following result which provides a class of bilateral generating matrix functions for the MJMPs and the Laguerre matrix polynomials.

Corollary 5.1. Let $\Lambda_{\mu,v}(y; z) := \sum_{k=0}^{\infty} a_k L_{\mu+vk}^{(C,\lambda)}(y) z^k$ where $(a_k \neq 0, \mu, v \in \mathbb{N}_0)$ and

$$\Theta_{\mathbf{n},p,\mu,v}(\mathbf{x}; y; \zeta) := \sum_{k=0}^{[n_1/p]} a_k P_{n_1-pk, n_2, \dots, n_s}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) L_{\mu+vk}^{(C,\lambda)}(y) \zeta^k$$

where $\mathbf{x} = (x_1, \dots, x_s)$, $\mathbf{A} = (A_1, \dots, A_s)$, $\mathbf{B} = (B_1, \dots, B_s)$, $\mathbf{n} = n_1 + \dots + n_s$; $n_1, \dots, n_s, p \in \mathbb{N}$, and A_i and B_i are matrices in $\mathbb{C}^{N \times N}$ satisfying spectral conditions $\operatorname{Re}(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $\operatorname{Re}(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$ for $1 \leq i \leq s$. Then we have

$$\begin{aligned} &\sum_{n_1, \dots, n_s=0}^{\infty} \Theta_{\mathbf{n},p,\mu,v}\left(\mathbf{x}; y; \frac{\eta}{t_1^p}\right) t_1^{n_1}, \dots, t_s^{n_s} \\ &= \prod_{i=1}^s F_4\left(I+B_i, I+A_i; I+A_i, I+B_i; \frac{(x_i-1)t_i}{2}, \frac{(x_i+1)t_i}{2}\right) \Lambda_{\mu,v}(y; \eta) \\ &\quad \times \left(\sqrt{\frac{(x_i-1)t_i}{2}} + \sqrt{\frac{(x_i+1)t_i}{2}} < 1; i = 1, 2, \dots, s\right) \end{aligned} \quad (5.2)$$

provided that each member of (5.2) exists.

Remark 5.1. Using the generating matrix function (5.1) for the Laguerre matrix polynomials and taking $a_k = 1$, $\mu = 0$, $v = 1$, we have

$$\begin{aligned} &\sum_{n_1, \dots, n_s=0}^{\infty} \sum_{k=0}^{[n_1/p]} P_{n_1-pk, n_2, \dots, n_s}^{(\mathbf{A}, \mathbf{B})}(\mathbf{x}) L_k^{(C,\lambda)}(y) \eta^k t_1^{n_1-pk} t_2^{n_2}, \dots, t_s^{n_s} \\ &= \prod_{i=1}^s F_4\left(I+B_i, I+A_i; I+A_i, I+B_i; \frac{(x_i-1)t_i}{2}, \frac{(x_i+1)t_i}{2}\right) (1-\eta)^{-(C+I)} \exp\left(\frac{-\lambda y \eta}{1-\eta}\right) \end{aligned}$$

where $|\eta| < 1$, $0 < y < \infty$ and $\sqrt{\frac{(x_i-1)t_i}{2}} + \sqrt{\frac{(x_i+1)t_i}{2}} < 1$ for $i = 1, \dots, s$.

Choose $r = 1$, and $\Omega_{\mu+\nu k}(y) = C_{\mu+\nu k}^D(y)$ where $\mu, \nu \in \mathbb{N}_0$ and D is a matrix in $\mathbb{C}^{N \times N}$ satisfying $\left(\frac{-z}{2}\right) \notin \sigma(D)$ for $\forall z \in \mathbb{Z}^+ \cup \{0\}$ in Theorem 4.2. Here, Gegenbauer matrix polynomials are generated by as follows [8]:

$$\sum_{n=0}^{\infty} C_n^D(y) t^n = (1 - 2yt + t^2)^{-D}, \quad |y| < 1. \quad (5.3)$$

Then we obtain the following bilateral generating matrix function for the MJMPs and Gegenbauer matrix polynomials.

Corollary 5.2. Let $\Lambda_{\mu,\nu}(y; z) := \sum_{k=0}^{\infty} a_k C_{\mu+\nu k}^D(y) z^k$ where $(a_k \neq 0, \mu, \nu \in \mathbb{N}_0)$ and

$$\Theta_{n,p,\mu,\nu}(\mathbf{x}; y; \zeta) := \sum_{k=0}^{\lfloor n/p \rfloor} a_k S_{n-pk}^{(A_1, \dots, A_s; B_1, \dots, B_s)}(\mathbf{x}) C_{\mu+\nu k}^D(y) \zeta^k$$

where $\mathbf{x} = (x_1, \dots, x_s)$, D is a matrix in $\mathbb{C}^{N \times N}$ satisfying $\left(\frac{-z}{2}\right) \notin \sigma(D)$ for $\forall z \in \mathbb{Z}^+ \cup \{0\}$ and A_i and B_i are matrices in $\mathbb{C}^{N \times N}$ satisfying the spectral conditions $\text{Re}(z_i) > -1$ for each eigenvalue $z_i \in \sigma(A_i)$ and $\text{Re}(\eta_i) > -1$ for each eigenvalue $\eta_i \in \sigma(B_i)$ for $i = 1, 2, \dots, s$. Then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu} \left(\mathbf{x}; y; \frac{\eta}{t^p} \right) t^n = \prod_{i=1}^s (1-t)^{-(A_i+B_i+I)} F \left(\frac{A_i+B_i+I}{2}, \frac{A_i+B_i+2I}{2}; A_i+I; \frac{2t(x_i-1)}{(1-t)^2} \right) \Lambda_{\mu,\nu}(y; \eta) \quad (5.4)$$

provided that each member of (5.4) exists.

Remark 5.2. Using (5.3) and taking $a_k = 1, \mu = 0, \nu = 1$, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/p \rfloor} S_{n-pk}^{(A_1, \dots, A_s; B_1, \dots, B_s)}(\mathbf{x}) C_k^D(y) \eta^k t^{n-pk} \\ &= \prod_{i=1}^s (1-t)^{-(A_i+B_i+I)} F \left(\frac{A_i+B_i+I}{2}, \frac{A_i+B_i+2I}{2}; A_i+I; \frac{2t(x_i-1)}{(1-t)^2} \right) (1-2y\eta + \eta^2)^{-D}. \end{aligned}$$

Setting $r = s$ and $\Omega_{\mu+\nu k}(y_1, \dots, y_s) = S_{\mu+\nu k}^{(C_1, \dots, C_s; D_1, \dots, D_s)}(y_1, \dots, y_s)$ in Theorem 4.2 where $\mu, \nu \in \mathbb{N}_0$, we obtain the following result which provides a class of bilinear generating matrix functions for the MJMPs.

Corollary 5.3. Let $\Lambda_{\mu,\nu}(y_1, \dots, y_s; z) := \sum_{k=0}^{\infty} a_k S_{\mu+\nu k}^{(C_1, \dots, C_s; D_1, \dots, D_s)}(y_1, \dots, y_s) z^k$ where $a_k \neq 0, \mu, \nu \in \mathbb{N}_0$ and

$$\Theta_{n,p,\mu,\nu}(\mathbf{x}; \mathbf{y}; \zeta) := \sum_{k=0}^{\lfloor n/p \rfloor} a_k S_{n-pk}^{(A_1, \dots, A_s; B_1, \dots, B_s)}(\mathbf{x}) S_{\mu+\nu k}^{(C_1, \dots, C_s; D_1, \dots, D_s)}(\mathbf{y}) \zeta^k \quad (n, p \in \mathbb{N})$$

where $\mathbf{x} = (x_1, \dots, x_s)$, $\mathbf{y} = (y_1, \dots, y_s)$, C_i and D_i are matrices in $\mathbb{C}^{N \times N}$ satisfying the spectral conditions $\text{Re}(\mu_i) > -1$ for each eigenvalue $\mu_i \in \sigma(C_i)$ and $\text{Re}(\nu_i) > -1$ for each eigenvalue $\nu_i \in \sigma(D_i)$ for $i = 1, 2, \dots, s$ and all matrices are commutative. Then we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu} \left(\mathbf{x}; \mathbf{y}; \frac{\eta}{t^p} \right) t^n \\ &= \prod_{i=1}^s (1-t)^{-(A_i+B_i+I)} F \left(\frac{A_i+B_i+I}{2}, \frac{A_i+B_i+2I}{2}; A_i+I; \frac{2t(x_i-1)}{(1-t)^2} \right) \Lambda_{\mu,\nu}(y_1, \dots, y_s; \eta) \end{aligned} \quad (5.5)$$

provided that each member of (5.5) exists.

Remark 5.3. Taking $a_k = 1, \mu = 0, \nu = 1$ and using Theorem 3.4, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/p \rfloor} S_{n-pk}^{(A_1, \dots, A_s; B_1, \dots, B_s)}(\mathbf{x}) S_k^{(C_1, \dots, C_s; D_1, \dots, D_s)}(\mathbf{y}) \eta^k t^{n-pk} \\ &= \prod_{i=1}^s (1-t)^{-(A_i+B_i+I)} F \left(\frac{A_i+B_i+I}{2}, \frac{A_i+B_i+2I}{2}; A_i+I; \frac{2t(x_i-1)}{(1-t)^2} \right) \\ & \quad \times \prod_{i=1}^s (1-\eta)^{-(C_i+D_i+I)} F \left(\frac{C_i+D_i+I}{2}, \frac{C_i+D_i+2I}{2}; C_i+I; \frac{2\eta(y_i-1)}{(1-\eta)^2} \right) \end{aligned}$$

where

$$S_n^{(A_1, \dots, A_s; B_1, \dots, B_s)}(x_1, \dots, x_s) = \sum_{n_1=0}^n \sum_{n_2=0}^{n-n_1} \cdots \sum_{n_{s-1}=0}^{n-n_1-\dots-n_{s-2}} \times \left\{ \mathbf{K}(n_1, \dots, n_{s-1}) P_{n-(n_1+\dots+n_{s-1}), n_1, \dots, n_{s-1}}^{(A_1, \dots, A_s; B_1, \dots, B_s)}(x_1, \dots, x_s) \mathbf{N}(n_1, \dots, n_{s-1}) \right\};$$

and

$$\mathbf{K}(n_1, \dots, n_{s-1}) = (A_1 + B_1 + I)_{n-(n_1+\dots+n_{s-1})} (A_2 + B_2 + I)_{n_1} \cdots (A_s + B_s + I)_{n_{s-1}},$$

$$\mathbf{N}(n_1, \dots, n_{s-1}) = [(A_s + I)_{n_{s-1}}]^{-1} [(A_{s-1} + I)_{n_{s-2}}]^{-1} \cdots [(A_2 + I)_{n_1}]^{-1} [(A_1 + I)_{n-(n_1+\dots+n_{s-1})}]^{-1}.$$

Furthermore, for every suitable choice of the coefficients a_k ($k \in \mathbb{N}_0$), if the multivariable function $\mathcal{Q}_{\mu+\nu k}(y_1, \dots, y_r)$, ($r \in \mathbb{N}$), is expressed as an appropriate product of several simpler functions, the assertions of Theorems 4.1 and 4.2 can be applied in order to derive various families of multilinear and multilateral generating matrix functions for the MJMPs.

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